Non-extensive Thermodynamics

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Content

• **Motivation:** q-phenomenology or extended thermodynamics; temperature fluctuations

• **Derivation:** from improved canonical treatment to Rényi and Tsallis formula

• **Application:** the physics of the reservoir: QGP in MIT bag, black hole, chaotic YM fields

• **Conclusion:** Physical interpretation of the fitted spectral temperature and power
Ideal Gas

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Content

- **Fundation:** cut power-law single-particle energy distribution; Gamma temperature fluctuations
- **Derivation:** from constant heat capacity eos to canonical Rényi and Tsallis formula
- **Discussion:** positive, negative, infinite and zero heat-capicity
- **Conclusion:** Physical interpretation of the power-law fitted spectral temperature and power
Constant heat capacity eos

• $C = \frac{dE}{dT} = C_0,$

• $T = T_0 + \frac{1}{C_0} E$

• $S = \int \frac{dE}{T} = S_0 + C_0 \ln \left( 1 + \frac{E}{C_0 T_0} \right)$

• $P = e^{-S(E)} = K_0 \left( 1 + \frac{E}{C_0 T_0} \right)^{-C_0}$  \text{thermodynamical probability } 1/W
Constant heat capacity eos

Probability of stand-alone „microcanonical”

- \[ P(E_1) = e^{-S(E_1)} = K_0 \left( 1 + \frac{E_1}{C_0 T_0} \right)^{-C_0} \]

**Conditional** probability of a subsystem of it:

- \[ P(E_1 | E - E_1) = \frac{P(E)}{P(E-E_1)} = e^{S(E-E_1) - S(E)} = \left( 1 - \frac{E_1}{C_0 T_0 + E} \right)^{C_0} \]
Constant heat capacity eos

Probability of stand-alone „microcanonical”

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Constant heat capacity eos

Two different subsystem-state probabilities:

• \( P(E_1) \neq P(E_1 | E - E_1) \)

**Because** entropy is not additive!

• \( S(E) \neq S(E_1) + S(E - E_1) \)
Constant heat capacity eos

Mutual information (based on joint and marginal prob.-s):

\[ I_{12} = \sum_{i,j} r_{ij} \log \frac{r_{ij}}{p_i q_j}, \quad p_i = \sum_j r_{ij}, \quad q_j = \sum_i r_{ij} \]

**Mutual information** (based on entropy)

- \[ I_S(E_1|E) = S(E_1) + S(E - E_1) - S(E) \]

**Zero mutual information** (based on additive formal log)

- \[ I_K(E_1|E) = K(S(E_1)) + K(S(E - E_1)) - K(S(E)) = 0 \]
Constant heat capacity eos

Factorizing „deformed thermodynamical probability“:

\[ P_K(E) = e^{-K(S(E))} \]

In this way (the part of an ideal gas is an ideal gas...)

- \[ P_K(E_1) = P_K(E_1|E - E_1) = e^{-K(S(E_1))} \]

From zero mutual information (based on additive formal log)

- \[ K(S(E_1)) = K(S(E)) - K(S(E - E_1)) \]
Constant heat capacity eos

Factorizing „deformed thermodynamical probability”:

\[ P(E) = e^{-K(S(E))} \]

In this way (the part of an ideal gas is an ideal gas…)

\[ P(E) = P(E_1) - K(S(E - E_1)) \]

Call it „improved canonical”
Constant heat capacity eos

Formal logarithm, alias "deformed entropy":

\[ K(S(E)) = \frac{e^{aS} - 1}{a} = \frac{E}{T_0}, \quad if \quad a = \frac{1}{C_0} \]

In this way (the part of an ideal gas is an ideal gas...)

- \( P_K(E) = e^{-K(S(E))} = e^{-E/T_0} \)

From the new entropy \( \Rightarrow \) factorizing subsystem

\[ P_K(E) = P_K(E_1) \cdot P_K(E - E_1) \]
Constant heat capacity eos

Its Gibbs ensemble average is „Tsallis-entropy“:

\[ K(S(E)) = \sum_i p_i K(-\ln p_i) = \frac{1}{a} \sum_i (p_i^{1-a} - p_i) \]

The original S functional is „Rényi-entropy“:

\[ S(E) = K^{-1}(\sum_i p_i K(-\ln p_i)) = \frac{1}{a} \ln \sum_i p_i^{1-a} \]
Constant heat capacity eos

Formal logarithm, alias „deformed entropy” for conditional prob.:

\[ K^*(S(E)) = \frac{1-e^{-aS}}{a} = \frac{E}{T}, \quad \text{if} \quad a = \frac{1}{c_0} \]

In this way (the part of an ideal gas is an ideal gas...)

- \( P_{K^*}(E) = e^{-K^*(S(E))} = e^{-E/T} \)

From the new entropy \( \rightarrow \) factorizing subsystem

\[ P_{K^*}(E) = P_{K^*}(E_1) \cdot P_{K^*}(E - E_1) \]
Constant heat capacity eos

Its Gibbs ensemble average is "Tsallis-entropy":

\[ K^*(S(E)) = \sum_i p_i K^*(-\ln p_i) = \frac{1}{a} \sum_i (p_i - p_i^{1+a}) \]

The original S functional is "Rényi-entropy":

\[ S(E) = K^{*-1}(\sum_i p_i K^*(-\ln p_i)) = \frac{1}{a} \ln \sum_i p_i^{1+a} \]
Ideal Gas: subsystem energy distributions (spectra)

\[ \ln P \]

\[ \mathcal{C}_0 > 0 \]

\[ P(E_1|E - E_1) \]

\[ P(E_1) \]

\[ P_K \text{ both} \]
Ideal Gas: subsystem energy distributions (spectra)

\[ P(E_1|E - E_1) \]

\[ P(E_1) \]

\[ P_K \text{ both} \]

\[ C_0 < 0 \]
Experiment: RHIC Star and Phenix data

Theoretical model: Tsallis quark matter + transverse flow + quark coalescence fits to hadron spectra

$\pi^0$

$K^+ + K^-$

$\eta$

$\phi$

Experiment: RHIC Star and Phenix data

Theoretical model: Tsallis quark matter + transverse flow + quark coalescence fits to hadron spectra

Not an exponential in $E_{\gamma}$!
Exp of a Log is Power-Law

\[ L(E) = \frac{1}{a} \ln(1 + aE) \]
Quark coalescence: weighted convolution

\[ \int \int \left( 1 + \beta \frac{E_1}{C} \right)^{-C} \left( 1 + \beta \frac{E_2}{C} \right)^{-C} \left| \varphi(E_1 - E_2) \right|^2 \delta(E_1 + E_2 - E) \]

\[ \approx \frac{1}{32} \left| \varphi(0) \right|^2 \left( 1 + \beta \frac{E}{2C} \right)^{-2C} \]

Meson = two quarks;

Quark coalescence: slope addition

Meson $q-1 = \text{quark} \ (q-1)/2$;
Baryon $q-1 = \text{quark} \ (q-1)/3$;

$$f_n \left( \frac{E}{n} \right) = \left( 1 + \beta \frac{E}{nC} \right)^{-nC}$$

Quark Coalescence

Meson = 2 quarks,  Baryon = 3 quarks

\[ P_{\text{meson}}(E) \approx P_{\text{quark}} \left( \frac{E}{2} \right) \cdot P_{\text{quark}} \left( \frac{E}{2} \right) \]

\[ P_{\text{baryon}}(E) \approx P_{\text{quark}} \left( \frac{E}{3} \right) \cdot P_{\text{quark}} \left( \frac{E}{3} \right) \cdot P_{\text{quark}} \left( \frac{E}{3} \right) \]

Consequence:

\[ \mathcal{I}_{\text{meson}}(E) = \mathcal{I}_{\text{quark}} \left( \frac{E}{2} \right) = T_0 + \frac{E}{2C}, \]

\[ \mathcal{I}_{\text{baryon}}(E) = \mathcal{I}_{\text{quark}} \left( \frac{E}{3} \right) = T_0 + \frac{E}{3C}, \]
Rising slopes with energy and mass

Meson $q-1 = \text{quark } (q-1)/2$;  
Baryon $q-1 = \text{quark } (q-1)/3$;  
Mini BH $q-1 = 2/\pi^2$

Power-Law Tails and Entropy Formulas

• Motivation: phenomenology and theory
  – Canonical Power-Law tailed Energy Spectra
  – Fits to AA, pp and e+e- (EPJA 40, 299, 2009; PLB 701, 111, 2011)
  – Rényi- and Tsallis (and more) Entropy Formulas, beta-fluctuations, superstatistics
  – Thermodynamics constraints possible formulas via allowed composition rules (PRE 83, 061147, 2011)
  – Scaled repetition leads to associative composition (EPL 84, 56003, 2008)
  – 2 body $\rightarrow$ maxprob $\rightarrow$ weighted sum of many copies: Gibbs’ ensemble average formula (arXiv 1208.2533, 1209.5963)
Entropy formulas

- $S = \ln \frac{N!}{\prod_i N_i!}$  
  Boltzmann (permutation)

- $S = - \sum P_i \ln P_i$  
  Gibbs (Planck)

- $S = \frac{1}{1-q} \ln \sum P_i^q$  
  Rényi

- $S = \frac{1}{q-1} \sum (P_i - P_i^q)$  
  Tsallis (Chravda, Aczél, Daróczy,...)

There are (much) more!
Canonical distribution with Rényi entropy

\[
\frac{1}{1-q} \ln \sum p_i^q - \alpha \sum p_i - \beta \sum p_i E_i = \max
\]

\[
\frac{1}{1-q} \frac{q p_i^{q-1}}{\sum p_i^q} = \alpha + \beta E_i
\]

This cut power-law distribution is an \textbf{excellent} fit to particle spectra in high-energy experiments!

\[
p_i = \frac{1}{e^{\hat{L}(S)}} \left( 1 + (1-q) \frac{\beta (E_i - \langle E \rangle)}{q} \right)^{-\frac{1}{q-1}}
\]
Canonical distribution with Tsallis entropy

\[
\frac{1}{1-q} \sum (p_i^q - p_i) - \alpha \sum p_i - \beta \sum p_i E_i = \text{max}
\]

\[
\frac{1}{1-q} q p_i^{q-1} = \alpha + \frac{1}{1-q} + \beta E_i
\]

\[
p_i = \left( Z^{1-q} + (1-q) \frac{\beta E_i}{q} \right)^{\frac{1}{q-1}}
\]

This cut power-law distribution is an excellent fit to particle spectra in high-energy experiments!
Superstatistics:

NBD = Euler ° Poisson

Power Law = Euler ° Gibbs

\[
\binom{-k-1}{n} (-f)^n (1+f)^{-k-1-n} = \int_0^\infty \frac{(x f)^n}{n!} e^{-f x} \cdot \frac{x^k}{k!} e^{-x} \, dx
\]

\[
\left(1 + \frac{\beta E}{k}\right)^{-k-1} = \int_0^\infty e^{-\frac{\beta E_i}{k} x} \cdot \frac{x^k}{k!} e^{-x} \, dx
\]

q = \frac{k}{k + 1}

Interpretation: event by event
multiplicity fluctuations;
Volume fluctuations;
Temperature fluctuations;
Why to use the Tsallis / Rényi entropy formulas?

• It generalizes the Boltzmann-Gibbs-Shannon formula
• It treats statistical entanglement between subsystem and reservoir (due to conservation)
• It claims to be universal (applicable for whatever material quality of the reservoir)
• It leads to a cut power-law energy distribution in the canonical treatment
Why not to use the Tsallis / Rényi entropy formulas?

• They lack 300 years of classical thermodynamic foundation
• Tsallis is not additive, Rényi is not linear
• There is an extra parameter $q$ (mysterious?)
• How do different $q$ systems equilibrate?
• Why this and not any other?
• It looks pretty much formal...
Derivation as improved canonical

- Derivation:
  - Microcanonical entropy maximum for two
  - Reservoir-independent temperature: the best one can (see also: Almeida, Physica A300, 424, 2001)
  - Which composition rule leads to higher order agreement (cannot be the simple addition)
  - Make the choice of the additive $L(S)$ universal $\rightarrow$ separation constant $= 1 / \text{heat capacity}$
  - Result: $L(S)$ is Tsallis entropy, $S$ is Rényi entropy
General Derivation: formulas

- Two bodies: \( K(S(E_1)) + K(S(E - E_1)) = \max \).

- Zeroth Law: \( \beta_1 = K'(S(E_1)) \cdot S'(E_1) \)

\[
= K'(S(E - E_1)) \cdot S'(E - E_1)
\]

Taylor: \( S(E - E_1) = S(E) - E_1 S'(E) + \frac{1}{2} E_1^2 S''(E) \) ...
Derivation: formulas

\[ \beta_1 = K'(S(E)) \cdot S'(E) \]

\[ - E_1 \left[ S'(E)^2 K''(S(E)) + S''(E)K'(S(E)) \right] \]

The content of the bracket be zero!
Derivation: formulas

\[ \beta = K'(S(E)) \cdot S'(E) \]

and the content of the bracket \([ \ ]\) is zero:

\[ \frac{K''(S)}{K'(S)} = \frac{-S''(E)}{S'(E)^2} = \frac{1}{C(E)} \]

Universal if constant:

\[ \frac{K'''(S)}{K'(S)} = \alpha \]
Derivation: formulas

The solution is:

\[ K(S) = \frac{e^{aS} - 1}{a} \]

This generates

\[ K(-\ln P_i) = \frac{1}{a} (P_i^{-a} - 1) \]
Derivation: formulas

• Generalized to an ensemble of subsystems:

\[ S = - \sum_i P_i \ln P_i \quad \rightarrow \quad K(S) = \sum_i P_i K(-\ln P_i) \]
Derivation: Tsallis entropy

The canonical principle becomes:

\[
\frac{1}{a} \sum \left( P_i^{1-a} - P_i \right) - \beta \sum P_i E_i - \alpha \sum P_i = \text{max}.
\]

The entropy with \( q = 1-a \)

\[
K(S) = S_{Tsallis} = \frac{1}{q-1} \sum (P_i - P_i^q).
\]
Derivation: Rényi entropy

The Rényi entropy is the original one,

but the Tsallis entropy is to be maximized canonically

\[
S_{\text{Rényi}} = K^{-1}(S_{\text{Tsallis}}) = \frac{1}{1-q} \ln \sum P_i^q
\]
Improved Canonical Distribution

• $P_i = \left( Z^{1-q} + (1 - q) \frac{\beta}{q} E_i \right)^{\frac{1}{q-1}}$

• Expressed by the reservoir’s physical parameters via using our results:

• $P_i = \frac{1}{Z} \left( 1 + \frac{Z^{-1/C}}{C-1} e^{S/C} \frac{1}{T} E_i \right)^{-C}$

Check infinite C limit!
Improved Canonical Distribution

- $P_i = \left( Z^{1-q} + (1 - q) \frac{\beta}{q} E_i \right)^{\frac{1}{q-1}}$

- Expressed by the reservoir’s physical parameters via using our results:

- $P_i = \frac{1}{Z} \left( 1 + \frac{Z^{-1/C}}{C-1} e^{S/C \frac{1}{T} E_i} \right)^{-C}$

Check infinite C limit!
Improved Canonical Distribution

- Slope of the energy distribution:
  \[ \frac{1}{\mathcal{Z}} = - \frac{d}{dE_i} \ln P_i, \quad \mathcal{Z} = T_0 + \frac{E_i}{C} \]

- Expressed by the reservoir’s physical parameters via using our results:
  \[ T_0 = T e^{-S/C} Z^{1/C} (1 - \frac{1}{C}) \]
  \[ \ln_q Z = C \left( Z^{1/C} - 1 \right) = K(S_1) - \frac{C}{C-1} \beta E_1 \]

Check infinite C limit!
Infinite heat capacity limit

- \( P_i \to \frac{1}{Z} e^{-E_i/T_{fit}} \) with

- \( T_{fit} = \frac{1}{\beta} = T \lim_{C \to \infty} e^{-S/C} \)
Finite subsystem corrections to infinite heat capacity limit

\[ T_1 = T \left( 1 + \frac{E_1}{CT} + \cdots \right) \quad \text{traditional S-expansion} \]

\[ T_1 = T e^{-S/C} \left( \frac{e^{S(E_1)/C}}{1 + 0 \cdot \frac{E_1}{CT} + \alpha \cdot \frac{E_1^2}{C^2 T^2} + \cdots} \right) \quad \text{Our expression} \]

Traditional: \( T_1 < T, \) falling in \( E_1; \)  Ours: \( T_1 < T, \) but rising in \( E_1 \)!
Gaussian approximation

- Deviations from $S=\text{max}$ equilibrium are traditionally considered as Gaussian:

$$P(\Delta E) = e^{S(E_1) + S(E - E_1 - \Delta E)} \approx e^{-S'(E - E_1) \Delta E + \frac{1}{2} S''(E - E_1) \Delta E^2} \approx e^{-\frac{1}{T} \Delta E - \frac{1}{2CT^2} \Delta E^2}$$
Gaussian approximation

• After Legendre transformation also $\beta$ fluctuates as Gaussian:

\[ P(\Delta \beta) \propto e^{-\frac{cT^2}{2}\Delta \beta^2} + \ldots \]

• Thermodynamic ”uncertainty” minimal
Gaussian approximation and beyond

Beta fluctuation

Particle spectra: $\langle e^{-\beta \omega} \rangle$
Application and Conclusion

• Application:
  – Reservoir = QGP at constant volume
  – Reservoir = QGP at constant pressure
  – Reservoir = QGP at constant entropy
  – Reservoir = classical Yang-Mills on lattice
  – Reservoir = (Schwarzschild) black hole

• Conclusion:
  – Why Tsallis / Rényi entropy ?
  – What is q ?
  – What is T ?
Heat capacity of QGP reservoir

- MIT bag model:

\[
E = V(\sigma T^4 + B), \quad p = \frac{\sigma T^4}{3} - B, \quad S = \frac{4\sigma VT^3}{3}
\]

\[
C = \frac{dE}{dT} = 4\sigma VT^3 + (\sigma T^4 + B) \frac{dV}{dT}
\]
Heat capacity of QGP reservoir

- MIT bag model:
  \[ E = V(\sigma T^4 + B), \quad p = \frac{\sigma T^4}{3} - B, \quad S = 4\sigma V T^3 / 3 \]

  \[ C_V = 4\sigma V T^3 = 3S, \quad C_p = \infty, \quad C_S = \frac{3}{4} S \left(1 - \frac{T^4}{T_0^4}\right) \]

V const. \[ T_{fit} = T \lim_{C \to \infty} e^{-S/C} = Te^{-1/3} \approx 0.7 T \]

P const. \[ T_{fit} = T \lim_{C \to \infty} e^{-S/C} = T \]

S const. \[ T_{fit} = T \lim_{C \to \infty} e^{-S/C} < Te^{-4/3} \approx 0.25 T \]
Heat capacity of QGP reservoir

- Chaotic classical Yang-Mills:
  \[ S(E) = C_0 \ln(1 + E/C_0 T_0), \]  
  constant heat capacity \( C \) !

  \[ T_{fit} = T_0, \quad T = T_0 + E/C_0 \]

- Schwarzschild black hole:
  \[ S = \alpha E^2, \quad \frac{1}{T} = 2\alpha E, \quad C = -2\alpha E^2 = -2S \]

  \[ T_{fit} = T \lim_{C \to \infty} e^{-S/C} = Te^{1/2} \approx 1.65 T \]
Fitted slopes

\[ T_1 = \frac{1}{\beta} = T \exp(-S/C); \quad T_{\text{slope}} = \text{gothic } T, \text{ fitted to experimental analysis} \]
Fitted slopes

\[ T_1 = \frac{1}{\beta} = T \exp(-S/C); \quad \text{Tslope} = \text{gothic } T, \text{ fitted to experimental analysis} \]
Universal Thermostat Independence

- **Improved Canonical Approach**: assumes statistical entanglement by using $K(S)$ optimized to $\frac{d^2 K(S(E))}{dE^2} = 0$, rendering finite size corrections to one order higher.

- Universally treats *finite heat capacity reservoirs* (but includes the infinite ones) by fixing $\frac{K''(S)}{K'(S)} = a = -\frac{S''(E)}{S'(E)^2}$.

- Tsallis entropy is $L(S)$, Rényi entropy is $S$.

- Fitted Boltzmann-Gibbs temperature may differ from that of the finite reservoir in this case: it explains lower $T$ fits by cut power-law.

- For QGP with $T = 175$ MeV one has a
  - i) $V=\text{const}$ fit $T = 125$ MeV,
  - ii) $S=\text{const}$ QGP around 50,
  - iii) Yang-Mills fit $T$ independent,
  - iv) mini BH fit $T = 288$ MeV too large.
Summary figure

1/C

BG

1/E
Summary figure

Physical point, found

Linear scaling: extensive
Summary figure

Physical point, found by fitting \( q \) to the best averages.

Linear scaling: extensive

Tsallis formula
Summary figure

Physical point, found by fitting $q$ to the best averages

1 / C

1-q

BG

1 / E

Linear scaling: extensive

Anomalous scaling: non-extensive

Tsallis formula
Summary figure

Physical point, found by fitting \( q \) to the best averages

A realistic reservoir model

Linear scaling: extensive

Anomalous scaling: non-extensive

Tsallis formula
Summary figure

Physical point, found by fitting $q$ to the best averages

Linear scaling: extensive

Anomalous scaling: non-extensive

A realistic reservoir model

Tsallis formula

Black hole

BG